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# Fixed Points of Automorphisms Permuting the Generators Cyclically in Free solvable Lie algebras

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**Abstract:** We investigate fixed points of an automorphism of a free solvable Lie algebra which permutes the generators cyclically. Let $\Theta$ be a cyclic permutation of order n which belongs to the nth symmetric group $S_n$. We give form of the fixed points of an automorphism of a free solvable Lie algebra which is induced by the permutation $\Theta$.

**Keywords:** Free Solvable Lie algebra, automorphism, fixed point, cyclic permutation.

## I.INTRODUCTION

Let F be the free Lie algebra freely generated by a set $X = \{x_1, \dots, x_n\}, n \geq 2$, over a field K. The derived series of F is defined as the following:

$\delta^0(F) = F, \delta^1(F) = F' = [F, F]$ and for $m > 1$ we define
$$\delta^m(F) = [\delta^{m-1}(F), \delta^{m-1}(F)].$$

Fixed points subalgebras of free Lie algebras are studied by Bryant[1] and Drensky [3]. In [2] Bryant and Papistas have obtained some results about fixed point subalgebras of relatively free Lie algebras. Later, Ekici and Sönmez [4] have given a criterion detecting nontrivial fixed points of IA- automorphisms of a free metabelian Lie algebra.

Fixed point subalgebras of automorphisms preserving the length of words of free solvable groups are described by Tomaszewski [5]. In this work we obtained corresponding results for free solvable Lie algebras.
By $L_m$ we denote the free solvable Lie algebra $F/_{\delta^m(F)}$ of rank n and solvability class m.

Let $\theta$ be an automorphism of F of order k induced by a permutation $\sigma \in S_n$, where $S_n$ is the nth symmetric group. The automorphism $\theta$ induces an automorphism $\bar{\theta}: L_m \to L_m$ which is defined as $\bar{\theta}(\bar{\omega}) = \theta(\omega) + \delta^m(F)$, where $\omega \in F$, $\bar{\omega} = \omega + \delta^m(F)$. For an element $\bar{\omega}$ of $L_m$ if $\bar{\theta}(\bar{\omega}) = \bar{\omega}$ then $\bar{\omega}$ is called a fixed point of L.
It can be easily seen that if $\theta$ has order k then every element of the form

$$\bar{\omega} + \bar{\theta}(\bar{\omega}) + \bar{\theta}^2(\bar{\omega}) + \dots + \bar{\theta}^{k-1} \quad (1)$$

is a fixed point for $\bar{\theta}$, where $k \geq 2$, $\bar{\omega} \in F/_{\delta^m(F)}$.

It is not obvious that only such elements are the fixed points. In this work we prove that every fixed point of $\bar{\theta}$ has the form (1).

## II. MAIN RESULT

Assume that $\sigma \in S_n$ is a product of disjoint cycles $\sigma_i$ of length $r_i$, $i = 1, \dots, s$. Let $G = F/_{F'}$.

Lemma
Assume that $\theta$ is an automorphism of F which is induced by $\sigma$. If $\hat{\theta}$ is an automorphism of G, induced by $\theta$ then every fixed points of $\hat{\theta}$ has the form

$$\sum_{i=1}^{s}\left(\hat{\omega}_i + \hat{\theta}(\hat{\omega}_i) + \hat{\theta}^2(\hat{\omega}_i) + \dots + \hat{\theta}^{r_i-1}(\hat{\omega}_i)\right),$$
where $\hat{\omega}_i = \beta_i \hat{x}_{i_1}, \beta_i \in K, i = 1, \dots, s$.

Proof
Let $\hat{\theta}$ be an automorphism of G, induced by $\theta$. If $\hat{v} \in G$ is a fixed point of $\hat{\theta}$ then $\hat{\theta}(\hat{v}) = \hat{v}$. The element $\hat{v}$ can be uniquely written as

$$\hat{v} = \sum_{j=1}^{n} c_j \hat{x}_j, \qquad c_j \in K$$

By taking into account the cycles of $\sigma$ we arrange the generators which we see in $\hat{v}$ as $\hat{v} = \sum_{i=1}^{s}\sum_{t=1}^{r_i} c_{i_t}\hat{x}_{i_t}$.
Using the equality $\hat{\theta}(\hat{v}) = \hat{v}$ we get
$$c_{i_t} = c_{i_l} = \beta_i, \quad 1 \leq t, l \leq r_i, \quad i = 1, \dots, s.$$
Therefore $\hat{v} = \sum_{i=1}^{s}\beta_i\left(\sum_{t=1}^{r_i}\hat{x}_{i_t}\right)$. Since $\hat{\theta}(\hat{x}_{i_1}) = x_{i_2}, \dots, \hat{\theta}\left(\hat{x}_{i_{r_i}}\right) = \hat{x}_{i_1}$ then

$$\sum_{t=1}^{r_i}\hat{x}_{i_t} = \left(I + \hat{\theta} + \hat{\theta}^2 + \dots + \hat{\theta}^{r_i-1}\right)(\hat{x}_{i_1})$$
and so $\hat{v} = \sum_{i=1}^{s}\left(I + \hat{\theta} + \hat{\theta}^2 + \dots + \hat{\theta}^{r_i-1}\right)(\beta_i\hat{x}_{i_1})$.■

By the above Lemma it is clear that if $\sigma$ is a cycle of order n and $\hat{\theta}$ is an automorphism of G induced by $\theta$ then every fixed point of $\hat{\theta}$ has the form
$$\hat{\omega} + \hat{\theta}(\hat{\omega}) + \hat{\theta}^2(\hat{\omega}) + \dots + \hat{\theta}^{n-1}(\hat{\omega}),$$
where $\hat{\omega} = \beta\hat{x}_k, \quad \beta_i \in K, \quad x_k \in X.$

Theorem

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Assume that $\sigma \in S_n$ be a cycle of order n and $\theta$ be an automorphism of F induced by $\sigma$.

If $\bar{\theta}$ is an automorphism of $L_m$ induced by $\theta$ then every fixed point of $\bar{\theta}$ has the form

$$\bar{\omega} + \bar{\theta}(\bar{\omega}) + \bar{\theta}^2(\bar{\omega}) + \ldots + \bar{\theta}^{n-1}(\bar{\omega}),$$

where $\bar{\omega} = \alpha\bar{x}_k + h$, $\alpha \in K$, $x_k \in X$, $h \in L'_m$.

Proof

Let $\bar{v} \in L_m$ be a fixed point of $\bar{\theta}$. We use induction on m. For $m = 1$ $L_1 = F/_{F'}$ is a free abelian Lie algebra. So by the Lemma the result is clear.

Suppose that the assertion is true for all positive integers less than m. Let $\tilde{\theta}$ be an automorphism of $L_{m-1}$ induced by $\theta$.

By induction hypothesis the fixed points of the automorphism $\tilde{\theta}$ of the algebra $L_{m-1} = F/_{\delta^{m-1}F}$ are the elements of the form

$$\tilde{\omega}_1 + \tilde{\theta}(\tilde{\omega}_1) + \tilde{\theta}^2(\tilde{\omega}_1) + \cdots + \tilde{\theta}^{n-1}(\tilde{\omega}_1),$$

where $\tilde{\omega}_1 = \alpha\tilde{x}_k + h_1$, $\alpha \in K$, $x_k \in X$, $h_1 \in L'_{m-1}$.

Let $\tilde{u}$ be a fixed point of $\tilde{\theta}$ in $L_{m-1}$. Assume that $\tilde{u} = \tilde{\psi}(\tilde{\omega}_1)$, where

$\tilde{\psi} = I + \tilde{\theta} + \tilde{\theta}^2 + \cdots + \tilde{\theta}^{n-1}$.

Since

$$L_{n,m-1} = F/_{\delta^{m-1}F} \cong (F/_{\delta^m F})/(\delta^{m-1}F/_{\delta^m F}),$$

then the preimage of $\tilde{u}$ in $F/_{\delta^m F}$ is of the form

$$a = \psi(\omega_1) + g + \delta^m F,$$

where $g \in \delta^{m-1}F/_{\delta^m F}$. Then we have $\bar{\theta}(\bar{g}) = \bar{g}$ in the algebra $\delta^{m-1}F/_{\delta^m F}$. By the Lemma the element $\bar{g}$ has the form

$$\bar{g} = \bar{\omega}_2 + \bar{\theta}(\bar{\omega}_2) + \bar{\theta}^2(\bar{\omega}_2) + \cdots + \bar{\theta}^{n-1}(\bar{\omega}_2), \quad (2)$$

where $\bar{\omega}_2 = \beta\bar{b}$, $\beta \in K$, $\bar{b} \in \delta^{m-1}F/_{\delta^m F}$. Hence

$$a = \psi(\omega_1 + \omega_2) + \delta^m F \quad (3)$$

Now let $\bar{v}$ be a fixed point of $\bar{\theta}$ in $L_m$. The element $\bar{v}$ can be written as $\bar{v} = \bar{v}_1 + \bar{v}_2$,

where $v_1 \in F(\mathrm{mod}\,\delta^{m-1}F)$, $v_2 \in \delta^{m-1}F$. Since $\bar{\theta}(\bar{v}) = \bar{v}$ we get $\tilde{\theta}(\tilde{v}_1) = \tilde{v}_1$ and

$\bar{\theta}(\bar{v}_2) = \bar{v}_2$. By (2) and (3) we see that $\bar{v}_1 + \bar{v}_2$ has the form

$$\bar{v}_1 + \bar{v}_2 = \bar{\psi}(\bar{\omega}_1 + \bar{\omega}_2), \text{ where } \bar{\omega}_1 = \alpha\bar{x}_k + h_1, \ \bar{\omega}_2 = \beta\bar{b}, \ \alpha, \beta \in K, h_1 \in L'_m, \bar{a} \in \delta^{m-1}F/_{\delta^m F}.$$

It can be easily seen that every element of the form

$$\bar{\omega} + \bar{\theta}(\bar{\omega}) + \bar{\theta}^2(\bar{\omega}) + \ldots + \bar{\theta}^{n-1}(\bar{\omega})$$

is a fixed point of $\bar{\theta}$.

## REFERENCES

[1] R.M. Bryant, On the fixed points of finite group acting on a free Lie algebra. J. London Math Soc., 43(2): 215-224, 1991.

[2] R.M. Bryant, A.I. Papistas, On the fixed points of a finite group acting on a relatively free Lie algebra, Glasg Math J.,42:167-181, 2000.

[3] V. Drensky, Fixed algebras of residually nilpotent Lie algebras. Proc Amer Math Soc.,120: 1021-1028, 1994.

[4] N.Ekici, D.Sönmez, Fixed points of IA- endomorphisms of a free metabelian Lie algebra. Proc Indian Acad Sci (Math Sci. ),121(4):405-416, 2011.

[5] T. Witold, Fixed points of automorphisms preserving the length of words in free solvable groups Arch. Math., 99:425-432, 2012.